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On Graded Bialgebra Deformations*

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Abstract. We introduce the graded bialgebra deformations, which explain the lifting method of Andruskiewitsch and Schneider. We also relate these graded bialgebra deformations with the corresponding graded bialgebra cohomology groups, which is the graded version of the one due to Gerstenhaber and Schack.

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1 Introduction

The classification of finite-dimensional pointed Hopf algebras is a basic problem in the theory of Hopf algebras. It is well known that any pointed Hopf algebra H has a coradical filtration, with respect to which one associates a coradically-graded Hopf algebra gr$H$. Following Andruskiewitsch and Schneider, the classification problem can be divided into two parts. One is the classification of all coradically-graded pointed Hopf algebras, and the other is to find all possible pointed Hopf algebras H with gr$H$ isomorphic to a given coradically-graded pointed Hopf algebra. The second part is just the lifting method in [1] and [2]. One of our motivations is to relate the lifting method with a certain bialgebra deformation theory.

The deformation theory for algebras was initiated by Gerstenhaber in [4], and its analogue for bialgebras appeared first in [5] (see also [6] and [10]). Inspired by the graded algebra deformation theory in [3] and [11], we develop in this paper the theory of graded bialgebra deformations and their corresponding cohomology groups. Moreover, this deformation theory can be used to explain the lifting method of Andruskiewitsch and Schneider.

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This paper is organized as follows. In Section 2, we recall the notion of liftings and introduce the graded bialgebra deformations, and we show that the lifting is just the same as the graded bialgebra deformation in the sense of Theorem 2.2. The graded-rigid bialgebras are also studied (Corollaries 2.3 and 2.4). In Section 3, we introduce the notion of graded "hat" bialgebra cohomology groups for graded bialgebras, which controls the graded bialgebra deformations (Theorem 3.3).

2 Liftings and Graded Bialgebra Deformations

We will work on a base field $\mathbb{K}$. All unadorned tensors are over $\mathbb{K}$. We refer to [13] for graded bialgebras and filtered bialgebras, and to [7] and [9] for graded linear maps.

Let us recall the lifting method of Andruskiewitsch and Schneider (for more details, see [2]). Note that the lifting defined here is a slight generalization.

Throughout this paper, $B = \oplus_{i\geq 0} B_{(i)}$ will be a graded bialgebra over $\mathbb{K}$ with identity element 1_B, multiplication map m, counit ε, and comultiplication Δ. Then B has a natural bialgebra filtration

$$B_0 \subseteq B_1 \subseteq \cdots \subseteq B_i \subseteq \cdots,$$

where $B_i = \oplus_{j\leq i} B_{(j)}$ for any $i \geq 0$.

A *lifting* of the graded bialgebra B is a filtered bialgebra structure, denoted by U, on the underlying filtered vector space B with the above filtration such that $\mathrm{gr}U = B$ as graded bialgebras, where $\mathrm{gr}U$ is the graded bialgebra associated to the filtered bialgebra U (see [13, p. 226]). (By $\mathrm{gr}U = B$, we use the natural identification of the underlying space $\mathrm{gr}U$ with B, i.e., $\mathrm{gr}U_{(i)} = B_i/B_{i-1} \cong B_{(i)}$ for each $i \geq 0$.)

For any lifting U of the graded bialgebra B, it follows from the definition that U and B have the same identity element and counit. Therefore, to give a lifting U, we just need to define the multiplication m_U and comultiplication Δ_U.

Two liftings U and V of the graded bialgebra B are said to be *equivalent* if there is a filtered bialgebra isomorphism $\theta : U \to V$ such that $\mathrm{gr}\theta = \mathrm{Id}_B$, where $\mathrm{gr}\theta$ is the graded morphism associated to θ, and here again we use the identifications $\mathrm{gr}U = B$ and $\mathrm{gr}V = B$ (as graded bialgebras).

Denote by $\mathrm{Lift}(B)$ the set of equivalent classes of all the liftings of the graded bialgebra B.

Let $l \in \mathbb{N} \cup \{+\infty\}$. Consider the space $B[t]/(t^{l+1})$, which is viewed as a free module over $\mathbb{K}[t]/(t^{l+1})$, and also a graded $\mathbb{K}$-space with $\deg t = 1$ and $\deg b = n$ if $b \in B_{(n)}$. If $l = +\infty$, then $B[t]/(t^{l+1})$ means $B[t]$ and $\mathbb{K}[t]/(t^{l+1})$ means $\mathbb{K}[t]$.

An *l-th level graded bialgebra deformation* of B consists of

$$m_t^l : (B \otimes B)[t]/(t^{l+1}) \to B[t]/(t^{l+1})$$

and

$$\Delta_t^l : B[t]/(t^{l+1}) \to (B \otimes B)[t]/(t^{l+1}) \cong B[t]/(t^{l+1}) \otimes_{\mathbb{K}[t]/(t^{l+1})} B[t]/(t^{l+1}),$$

which are $\mathbb{K}[t]/(t^{l+1})$-linear and homogeneous maps of degree zero such that

(i) $B[t]/(t^{l+1})$ is a bialgebra over $\mathbb{K}[t]/(t^{l+1})$ with identity element 1_B, multiplication m_t^l, counit ε_t^l and comultiplication Δ_t^l, where the counit $\varepsilon_t^l : B[t]/(t^{l+1}) \to \mathbb{K}[t]/(t^{l+1})$ is given by $\varepsilon_t^l(bt^j) = \varepsilon(b)t^j$ for $b \in B$ and $0 \le j \le l$;

(ii) $m_t^l \equiv m \otimes \mathrm{Id}_{\mathbb{K}[t]/(t^{l+1})}$ and $\Delta_t^l \equiv \Delta \otimes \mathrm{Id}_{\mathbb{K}[t]/(t^{l+1})}$ mod (t), where m and Δ are the multiplication and comultiplication of B, respectively.

Denote by $(B[t]/(t^{l+1}), m_t^l, \Delta_t^l)$ the above l-th level graded bialgebra deformation.

From now on, we will abbreviate l-th level graded bialgebra deformations as l-deformations, and $+\infty$-deformations will be referred simply as deformations. Denote by $\mathcal{E}^l(B)$ the set of all l-deformations of the graded bialgebra B, and $\mathcal{E}^{+\infty}(B)$ is written as $\mathcal{E}(B)$. Elements of $\mathcal{E}(B)$ will be written as the form $(B[t], m_t, \Delta_t)$.

Two l-deformations $(B[t]/(t^{l+1}), m_t^l, \Delta_t^l)$ and $(B[t]/(t^{l+1}), {m'}_t^l, {\Delta'}_t^l)$ are said to be *isomorphic* if there exists an isomorphism of $\mathbb{K}[t]/(t^{l+1})$-bialgebras

$$\phi : (B[t]/(t^{l+1}), m_t^l, \Delta_t^l) \to (B[t]/(t^{l+1}), {m'}_t^l, {\Delta'}_t^l)$$

such that ϕ is homogeneous of degree zero and $\phi \equiv \mathrm{Id}_B \otimes \mathrm{Id}_{\mathbb{K}[t]/(t^{l+1})}$ mod (t).

Denote by $\mathrm{iso}\,\mathcal{E}^l(B)$ (resp., $\mathrm{iso}\,\mathcal{E}(B)$) the set of isoclasses of l-deformations (resp., deformations) of the graded bialgebra B for $l \in \mathbb{N}$.

Consider an element $(B[t]/(t^{l+1}), m_t^l, \Delta_t^l)$ of $\mathcal{E}^l(B)$. By definition, we can write

$$m_t^l(a \otimes b) = \sum_{0 \le s \le l} m_s(a \otimes b)t^s \tag{1}$$

and

$$\Delta_t^l(c) = \sum_{0 \le s \le l} \Delta_s(c)t^s, \tag{2}$$

where $a, b, c \in B$, and $m_s : B \otimes B \to B$ and $\Delta_s : B \to B \otimes B$ are homogeneous of degree $-s$. Note that $m_0 = m$ and $\Delta_0 = \Delta$.

It is easy to check that the associativity of m_t^l, the compatibility of m_t^l and Δ_t^l, and the coassociativity of Δ_t^l are respectively equivalent to the following identities for each $1 \le n \le l$:

$$\begin{aligned} &am_n(b \otimes c) - m_n(ab \otimes c) + m_n(a \otimes bc) - m_n(a \otimes b)c \\ &= \sum_{1 \le s \le n-1} m_s(m_{n-s}(a \otimes b) \otimes c) - m_s(a \otimes m_{n-s}(b \otimes c)), \end{aligned} \tag{3}$$

$$\begin{aligned} &m_n(a_{(1)} \otimes b_{(1)}) \otimes a_{(2)}b_{(2)} - \Delta(m_n(a \otimes b)) + a_{(1)}b_{(1)} \otimes m_n(a_{(2)} \otimes b_{(2)}) \\ &+ a_{(1)}b_l \otimes a_{(2)}b_r - \Delta_n(ab) + a_l b_{(1)} \otimes a_r b_{(2)} \\ &= - \sum_{0 \le s,r,s',r' \le n-1,\ s+s'+r+r'=n} (m_r \otimes m_{r'}) \circ \tau_{23} \circ (\Delta_s \otimes \Delta_{s'})(a \otimes b) \\ &+ \sum_{1 \le s \le n-1} \Delta_s(m_{n-s}(a \otimes b)), \end{aligned} \tag{4}$$

and

$$\begin{aligned} &c_{(1)} \otimes \Delta_n(c_{(2)}) - (\Delta \otimes \mathrm{Id}) \circ \Delta_n(c) + (\mathrm{Id} \otimes \Delta) \circ \Delta_n(c) - \Delta_n(c_{(1)}) \otimes c_{(2)} \\ &= \sum_{1 \le s \le n-1} (\Delta_{n-s} \otimes \mathrm{Id}) \circ \Delta_s(c) - (\mathrm{Id} \otimes \Delta_{n-s}) \circ \Delta_s(c), \end{aligned} \tag{5}$$

where we use Sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for $a \in B$, in the second identity we use the notation $\Delta_n(a) = a_l \otimes a_r$ and $\Delta_n(b) = b_l \otimes b_r$, and the map τ_{23} is the canonical flip map at the second and third positions.

Let $(B[t]/(t^{l+1}), m_t^l, \Delta_t^l)$ and $(B[t]/(t^{l+1}), {m'}_t^l, {\Delta'}_t^l)$ be two l-deformations with the maps m_s, Δ_s and m'_s, Δ'_s as in (1) and (2). An isomorphism ϕ between these deformations is given by

$$\phi(a) = \sum_{0 \leq s \leq l} \phi_s(a) t^s, \quad a \in B,$$

where $\phi_s : B \to B$ is a homogeneous map of degree $-s$. Note that $\phi_0 = \mathrm{Id}_B$. The fact that ϕ is a morphism of $\mathbb{K}[t]/(t^{l+1})$-bialgebras implies that ϕ preserves the identity element 1_B and counit ε_t^l, and for each $1 \leq n \leq l$,

$$\begin{aligned} & (m_n - m'_n)(a \otimes b) \\ = \ & a\phi_n(b) - \phi_n(ab) + \phi_n(a)b \\ & + \sum_{0<s<n} \big\{ \phi_s(a)\phi_{n-s}(b) - \phi_s(m_{n-s}(a \otimes b)) + \sum_{r+r'=n-s} m'_s(\phi_r(a) \otimes \phi_{r'}(b)) \big\} \end{aligned}$$

and

$$\begin{aligned} & (\Delta_n - \Delta'_n)(c) \\ = \ & \Delta(\phi_n(c)) - c_{(1)} \otimes \phi_n(c_{(2)}) - \phi_n(c_{(1)}) \otimes c_{(2)} \\ & + \sum_{0<s<n} \big\{ \Delta'_s(\phi_{n-s}(c)) - (\phi_s \otimes \phi_{n-s})(\Delta(c)) - \sum_{r+r'=n-s} (\phi_r \otimes \phi_{r'})(\Delta_s(c)) \big\} \end{aligned}$$

for all $a, b, c \in B$. Note that the above discussion works for all $l \in \mathbb{N} \cup \{+\infty\}$.

The analogue of the following lemma is well known in the classical deformation theory.

Lemma 2.1. *There exist restriction maps $r_{l,l'} : \mathcal{E}^l(B) \to \mathcal{E}^{l'}(B)$ for any $l > l' \in \mathbb{N}$ and maps $r_l : \mathcal{E}(B) \to \mathcal{E}^l(B)$ such that $\mathcal{E}(B) = \varprojlim_{l \in \mathbb{N}} \mathcal{E}^l(B)$.*

Proof. Given $(B[t]/(t^{l+1}), m_t^l, \Delta_t^l)$ in $\mathcal{E}^l(B)$ with maps m_s and Δ_s as in (1) and (2), define $m_t^{l'} := \sum_{0 \leq s \leq l'} m_s t^s$ and $\Delta_t^{l'} := \sum_{0 \leq s \leq l'} \Delta_s t^s$; it is direct to check that $(B[t]/(t^{l'+1}), m_t^{l'}, \Delta_t^{l'})$ is the desired element in $\mathcal{E}^{l'}(B)$. The map r_l is defined in a similar way, and then the result is obvious. □

A graded bialgebra $B = \oplus_{i \geq 0} B_{(i)}$ is called *graded-rigid* if the set $\mathrm{iso}\,\mathcal{E}(B)$ has only one element, i.e., any deformation of B is isomorphic to the trivial one.

We have the following observation, which says that the graded bialgebra deformations coincide with the liftings.

Theorem 2.2. *Let $B = \oplus_{i \geq 0} B_{(i)}$ be a graded bialgebra. There exists a natural bijection $\mathrm{Lift}(B) \simeq \mathrm{iso}\,\mathcal{E}(B)$.*

Proof. We will construct a map $F : \mathrm{Lift}(B) \to \mathrm{iso}\,\mathcal{E}(B)$. Given a lifting U of B, denote by m_U and Δ_U the multiplication and comultiplication maps of U, respectively.

Since U is a filtered bialgebra, we have

$$m_U : B_i \otimes B_j \to B_{i+j} \quad \text{and} \quad \Delta_U : B_n \to \sum_{i+j=n} B_i \otimes B_j.$$

Therefore, for any $s \geq 0$, there uniquely exist homogeneous maps of degree $-s$, say $m_s : B \otimes B \to B$ and $\Delta_s : B \to B \otimes B$ such that

$$m_U(a \otimes b) = \sum_{s \geq 0} m_s(a \otimes b) \quad \text{and} \quad \Delta_U(c) = \sum_{s \geq 0} \Delta_s(c).$$

Since $\mathrm{gr}U = B$ as graded bialgebras, we have $m_0 = m$ and $\Delta_0 = \Delta$.

Now define $F(U) = (B[t], m_t, \Delta_t)$ as follows:

$$m_t(a \otimes b) := \sum_{s \geq 0} m_s(a \otimes b) t^s \quad \text{and} \quad \Delta_t(c) := \sum_{s \geq 0} \Delta_s(c) t^s.$$

It is direct to check that $F(U)$ is a deformation.

The map F is well defined, i.e., it maps equivalent liftings to isomorphic deformations. In fact, for given liftings U and V, an equivalence θ of U and V is a filtered isomorphism, hence for any $s \geq 0$, it determines a unique homogeneous map $\phi_s : B \to B$ of degree $-s$ such that $\theta(a) = \sum_{s \geq 0} \phi_s(a)$ for $a \in B$. Then we can define a $\mathbb{K}[t]$-linear map $\phi : B[t] \to B[t]$ such that $\phi(a) = \sum_{s \geq 0} \phi_s(a) t^s$. Hence, ϕ is an isomorphism between the deformations $F(U)$ and $F(V)$.

By (1) and (2), one obtains that F is a bijection. This completes the proof. $\square$

An immediate consequence of Theorem 2.2 is:

Corollary 2.3. *Let $B = \oplus_{i \geq 0} B_{(i)}$ be a graded bialgebra. If B is graded-rigid, then for any filtered bialgebra U such that* $\mathrm{gr}U \simeq B$ *as graded bialgebras, we have $U \simeq B$ as bialgebras. If we assume B is coradically-graded, then the converse is also true.*

Proof. By Theorem 2.2, B is graded-rigid if and only if $\mathrm{Lift}(B)$ is a single-element set, i.e., every lifting of B is trivial.

For the first statement, such a filtered bialgebra U with $\mathrm{gr}U \simeq B$ gives rise to a lifting on B, denoted by U', such that $U \simeq U'$ (as bialgebras). Since B is graded-rigid, we get $U' \simeq B$, thus we are done.

For the second one, assume B is coradically-graded. Let U be a lifting of B. By assumption, there exists an isomorphism $\theta : U \simeq B$. Note that θ preserves the coradical filtration, thus $\mathrm{gr}\theta$ can be viewed as a graded automorphism of B. Take $\theta' = (\mathrm{gr}\theta)^{-1} \circ \theta : U \simeq B$. Then θ' realizes an equivalence between the lifting U and the trivial lifting. This proves that B is graded-rigid. $\square$

We assume the base field $\mathbb{K}$ is algebraically closed of characteristic zero. One can define the variety Bialg_n of the bialgebra structures on n-dimensional spaces, which carries a natural $GL_n(\mathbb{K})$-action by base changes (see [8] and [12]). Recall that a bialgebra B is called rigid if the $GL_n(\mathbb{K})$-orbit of Bialg_n containing B is Zariski open. In fact, we have:

Corollary 2.4. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and $B = \oplus_{i\geq 0} B_{(i)}$ a finite-dimensional graded bialgebra over $\mathbb{K}$. If B is rigid and coradically-graded, then B is graded-rigid.*

Proof. By Corollary 2.3, we only need to show that every filtered bialgebra U with $\mathrm{gr}U \simeq B$ is isomorphic to B. Assume the dimension of B is n. By Theorem 3.4 in [8], B is a degeneration of U, i.e., it lies in the closure of the orbit of U (in the variety Bialg_n). However, the $GL_n(\mathbb{K})$-orbit of B is open, we obtain that B and U belong to the same $GL_n(\mathbb{K})$-orbit, i.e., $B \simeq U$ as bialgebras, finishing the proof. □

3 Graded Bialgebra Cohomology

In this section, we will relate the graded bialgebra deformations with the corresponding cohomology groups, which will be a graded (and normalized) version of "hat" bialgebra cohomology groups introduced in [5] (see also [10]).

Let $(B, m, e, \Delta, \varepsilon)$ be a bialgebra.

Let us recall the bicomplex in [5] or [10, p. 619]. For $p, q \geq 1$, let the maps $\lambda^p : B^{\otimes p+1} \to B^{\otimes p}$ and $\rho^p : B^{\otimes p+1} \to B^{\otimes p}$ be given by

$$\lambda^p(b^1 \otimes \cdots \otimes b^{p+1}) = b^1_{(1)} b^2 \otimes \cdots \otimes b^1_{(p)} b^{p+1},$$
$$\rho^p(b^1 \otimes \cdots \otimes b^{p+1}) = b^1 b^{p+1}_{(1)} \otimes \cdots \otimes b^p b^{p+1}_{(p)}.$$

Dually, the maps $\sigma^q : B^{\otimes q} \to B^{\otimes q+1}$ and $\tau^q : B^{\otimes q} \to B^{\otimes q+1}$ are given by

$$\sigma^q(b^1 \otimes \cdots \otimes b^q) = (b^1_{(1)} \cdots b^q_{(1)}) \otimes b^1_{(2)} \otimes \cdots b^q_{(2)},$$
$$\tau^q(b^1 \otimes \cdots \otimes b^q) = b^1_{(1)} \otimes \cdots \otimes b^q_{(1)} \otimes (b^1_{(2)} \cdots b^q_{(2)}).$$

In addition, we define $\Delta^p_i : B^{\otimes p} \to B^{\otimes p+1}$ and $\mu^q_j : B^{\otimes q+1} \to B^{\otimes q}$ for $1 \leq i \leq p$ and $1 \leq j \leq q$ by

$$\Delta^p_i(b^1 \otimes \cdots \otimes b^p) = b^1 \otimes \cdots \otimes b^i_{(1)} \otimes b^i_{(2)} \otimes \cdots \otimes b^p,$$
$$\mu^q_i(b^1 \otimes \cdots \otimes b^{q+1}) = b^1 \otimes \cdots \otimes b^i b^{i+1} \otimes \cdots \otimes b^{q+1}.$$

Let $C^{p,q} = \mathrm{Hom}_{\mathbb{K}}(B^{\otimes q}, B^{\otimes p})$, $p, q \geq 1$. Define

$$\delta^{p,q}_h : C^{p,q} \to C^{p,q+1} \quad \text{and} \quad \delta^{p,q}_c : C^{p,q} \to C^{p+1,q}$$

by

$$\delta^{p,q}_h(f) = \lambda^p \circ (\mathrm{Id} \otimes f) + \sum_{i=1}^{q} (-1)^i f \circ \mu^q_i + (-1)^{q+1} \rho^p \circ (f \otimes \mathrm{Id}),$$
$$\delta^{p,q}_c(f) = (\mathrm{Id} \otimes f) \circ \sigma^q + \sum_{j=1}^{p} (-1)^j \Delta^p_j \circ f + (-1)^{p+1} (f \otimes \mathrm{Id}) \circ \tau^q$$

for $f \in C^{p,q}$, where Id denotes the identity map of B.

It is direct to check that $(C^{p,q}, \delta^{p,q}_h, \delta^{p,q}_c)$ is a bicomplex (see [10, p. 619]), i.e.,

$$\delta^{p,q+1}_h \circ \delta^{p,q}_h = 0, \quad \delta^{p,q+1}_c \circ \delta^{p,q}_h = \delta^{p+1,q}_h \circ \delta^{p,q}_c, \quad \delta^{p+1,q}_c \circ \delta^{p,q}_c = 0.$$

We will introduce a sub-bicomplex of the above bicomplex. Let $\mathsf{m} = \operatorname{Ker}\varepsilon$. Denote by $i : \mathsf{m} \to B$ the inclusion map, and $\pi\colon B \to \mathsf{m}$ is given by $\pi(b) = b - \varepsilon(b)1_B$. Set $D^{p,q} = \operatorname{Hom}_{\mathbb{K}}(\mathsf{m}^{\otimes q}, \mathsf{m}^{\otimes p})$, $p, q \geq 1$. Note that we have a natural embedding $D^{p,q} \hookrightarrow C^{p,q}$ by identifying $f \in D^{p,q}$ with $i^{\otimes p} \circ f \circ \pi^{\otimes q} \in C^{p,q}$.

Lemma 3.1. *We have* $\delta_h^{p,q}(D^{p,q}) \subseteq D^{p,q+1}$ *and* $\delta_c^{p,q}(D^{p,q}) \subseteq D^{p+1,q}$.

Proof. Just note that $f \in C^{p,q}$ lies in $D^{p,q}$ if and only if

$$(\operatorname{Id}^{\otimes j-1} \otimes \varepsilon \otimes \operatorname{Id}^{\otimes p-j}) \circ f = 0$$

and

$$f(b^1 \otimes \cdots \otimes b^{i-1} \otimes 1 \otimes b^{i+1} \otimes \cdots \otimes b^q) = 0$$

for any $1 \leq i \leq q$, $1 \leq j \leq p$ and any $b^i \in B$. Then the lemma follows from the definition of $\delta_h^{p,q}$ and $\delta_c^{p,q}$ immediately. □

From now on, $B = \oplus_{i \geq 0} B_{(i)}$ will be a graded bialgebra. In this case, $\mathsf{m} \subseteq B$ is a graded subspace. Consider $D_{(l)}^{p,q} := \operatorname{Hom}_{\mathbb{K}}(\mathsf{m}^{\otimes q}, \mathsf{m}^{\otimes p})_{(l)}$, $l \in \mathbb{Z}$, whose elements are homogeneous maps from $\mathsf{m}^{\otimes q}$ to $\mathsf{m}^{\otimes p}$ of degree l. Note that $D_{(l)}^{p,q} \subseteq D^{p,q} \hookrightarrow C^{p,q}$.

Lemma 3.2. *We have* $\delta_h^{p,q}(D_{(l)}^{p,q}) \subseteq D_{(l)}^{p,q+1}$ *and* $\delta_c^{p,q}(D_{(l)}^{p,q}) \subseteq D_{(l)}^{p+1,q}$ *for any* $l \in \mathbb{Z}$ *and* $p, q \geq 1$.

Proof. Set $C_{(l)}^{p,q} = \operatorname{Hom}_{\mathbb{K}}(B^{\otimes q}, B^{\otimes p})_{(l)}$. Clearly, $D_{(l)}^{p,q} = D^{p,q} \cap C_{(l)}^{p,q}$. From the definition of $\delta_h^{p,q}$ and $\delta_c^{p,q}$, one sees that they preserve the degree, i.e., $\delta_h^{p,q}(C_{(l)}^{p,q}) \subseteq C_{(l)}^{p,q+1}$ and $\delta_c^{p,q}(C_{(l)}^{p,q}) \subseteq C_{(l)}^{p+1,q}$. Now the result follows from Lemma 3.1. □

Denote by $\delta_{h,(l)}^{p,q}$ (resp., $\delta_{c,(l)}^{p,q}$) the restriction of the map $\delta_h^{p,q}$ (resp., $\delta_c^{p,q}$) to the subspace $D_{(l)}^{p,q}$. Thus, by Lemma 3.2, we get a bicomplex $(D_{(l)}^{p,q}, \delta_{h,(l)}^{p,q}, \delta_{c,(l)}^{p,q})$ for each $l \in \mathbb{Z}$.

There is a canonical way to construct a complex from a given bicomplex. For $n \geq 1$, set

$$\hat{D}_{(l)}^n = \bigoplus_{p+q=n+1,\ p,q \geq 1} D_{(l)}^{p,q},$$

and define $\partial_{(l)}^n : \hat{D}_{(l)}^n \to \hat{D}_{(l)}^{n+1}$ by $\partial_{(l)}^n|_{D_{(l)}^{n+1-q,q}} := \delta_{h,(l)}^{p,q} + (-1)^q \delta_{c,(l)}^{p,q}$ for $1 \leq q \leq n$. Hence, for each $l \in \mathbb{Z}$, we get a complex

$$0 \to \hat{D}_{(l)}^1 \xrightarrow{\partial_{(l)}^1} \hat{D}_{(l)}^2 \xrightarrow{\partial_{(l)}^2} \hat{D}_{(l)}^3 \xrightarrow{\partial_{(l)}^3} \hat{D}_{(l)}^4 \to \cdots .$$

For $n \geq 1$, we define the n-th cohomology group of the above complex to be the *n-th graded "hat" bialgebra cohomology of degree l* of the graded bialgebra B, which will be denoted by $\hat{h}_b^n(B)_{(l)}$.

It is very useful to write out $\hat{h}_b^2(B)_{(l)}$ and $\hat{h}_b^3(B)_{(l)}$ explicitly from the definition. In what follows, we will use $\delta_h^{p,q}$ and $\delta_c^{p,q}$ instead of $\delta_{h,(l)}^{p,q}$ and $\delta_{c,(l)}^{p,q}$ for simplicity. We have the following facts:

The cohomology group $\hat{h}^2_b(B)_{(l)}$ consists of all pairs (f, g), where $f : \mathsf{m} \otimes \mathsf{m} \to \mathsf{m}$ and $g : \mathsf{m} \to \mathsf{m} \otimes \mathsf{m}$ are homogeneous maps of degree l, satisfying the following relations:

$$\delta_h^{1,2}(f) = 0, \quad \delta_c^{1,2}(f) + \delta_h^{2,1}(g) = 0, \quad \delta_c^{2,1}(g) = 0,$$

i.e., for any $a, b, c \in \mathsf{m}$, we have

$$af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c = 0, \tag{6}$$

$$\begin{aligned} f(a_{(1)} \otimes b_{(1)}) \otimes a_{(2)}b_{(2)} - \Delta(f(a \otimes b)) + a_{(1)}b_{(1)} \otimes f(a_{(2)} \otimes b_{(2)}) \\ + a_{(1)}g(b)_l \otimes a_{(2)}g(b)_r - g(ab) + g(a)_l b_{(1)} \otimes g(a)_r b_{(2)} = 0, \end{aligned} \tag{7}$$

$$c_{(1)} \otimes g(c_{(2)}) - (\Delta \otimes \mathrm{Id})(g(c)) + (\mathrm{Id} \otimes \Delta)(g(c)) - g(c_{(1)}) \otimes c_{(2)} = 0, \tag{8}$$

where we write $g(b) = g(b)_l \otimes g(b)_r$ for $b \in B$.

Two pairs (f, g) and (f', g') are equal in $\hat{h}^2_b(B)_{(l)}$ if and only if there exists a homogeneous map $\theta : \mathsf{m} \to \mathsf{m}$ of degree l such that for any $a, b, c \in \mathsf{m}$,

$$(f - f')(a \otimes b) = a\theta(b) - \theta(ab) + \theta(a)b, \tag{9}$$

$$(g - g')(c) = \Delta(\theta(c)) - c_{(1)} \otimes \theta(c_{(2)}) - \theta(c_{(1)}) \otimes c_{(2)}. \tag{10}$$

The group $\hat{h}^3_b(B)_{(l)}$ consists of all triples (F, H, G), where $F : \mathsf{m} \otimes \mathsf{m} \otimes \mathsf{m} \to \mathsf{m}$, $H : \mathsf{m} \otimes \mathsf{m} \to \mathsf{m} \otimes \mathsf{m}$, $G : \mathsf{m} \to \mathsf{m} \otimes \mathsf{m} \otimes \mathsf{m}$ are homogeneous maps of degree l, subject to the relations

$$\delta_h^{1,3}(F) = 0, \quad \delta_h^{2,2}(F) = \delta_c^{1,3}(H), \quad \delta_c^{2,2}(H) = -\delta_h^{1,3}(G), \quad \delta_c^{3,1}(G) = 0.$$

Note that $(F, H, G) = 0$ in $\hat{h}^3_b(B)_{(l)}$ if and only if there exists $(f, g) \in \hat{D}^2_{(l)}$ such that

$$(F, H, G) = \partial^2_{(l)}((f, g)), \tag{11}$$

which can be written out explicitly by the definition of $\partial^2_{(l)}$.

Now we are in the position to present our main observations, which relate the graded bialgebra deformations of the graded bialgebra B with the cohomology groups $\hat{h}^2_b(B)_{(l)}$ and $\hat{h}^3_b(B)_{(l)}$ (compare [5, Section 5]).

Theorem 3.3. *Let $B = \oplus_{i \geq 0} B_{(i)}$ be a graded bialgebra.*

(i) *There is a bijection between* $\mathrm{iso}\,\mathcal{E}^1(B)$ *and* $\hat{h}^2_b(B)_{(-1)}$.

(ii) *If $\hat{h}^2_b(B)_{(-l)} = 0$ for each $l \geq 1$, then the graded bialgebra B is graded-rigid.*

(iii) *For $l \geq 1$, the obstruction to extend an element of $\mathcal{E}^l(B)$ to $\mathcal{E}^{l+1}(B)$ lies in $\hat{h}^3_b(B)_{(-l-1)}$. In particular, if $\hat{h}^3_b(B)_{(-l-1)} = 0$, one can extend any element of $\mathcal{E}^l(B)$ to $\mathcal{E}^{l+1}(B)$.*

Proof. (i) Recall that an element in $\mathcal{E}^1(B)$ is of the form $(B[t]/(t^2), m_t^1, \Delta_t^1)$. Write

$$m_t^1(a \otimes b) = ab + f(a \otimes b)t \quad \text{and} \quad \Delta_t^1(c) = \Delta(c) + g(c)t,$$

where $f : B \otimes B \to B$ and $g : B \to B \otimes B$ are homogeneous of degree -1. Note that 1_B is the identity element of $B[t]/(t^2)$, hence $f(1_B \otimes b) = f(b \otimes 1_B) = 0$ for all $b \in B$. Moreover, for $a, b \in \mathsf{m}$, $\varepsilon_t^1(m_t^1(a \otimes b)) = 0$ implies $\varepsilon_t^1(ab + f(a \otimes b)t) = 0$, i.e., $f(a \otimes b) \in \mathsf{m}$. Thus, we may view f as in $D^{1,2}_{(-1)}$. Dually, one can show $g \in D^{2,1}_{(-1)}$.

Note that m_t^1 is an associative multiplication on $B[t]/(t^2)$, thus we get

$$f(a \otimes b)c - f(a \otimes bc) + f(ab \otimes c) - af(b \otimes c) = 0$$

for all $a, b, c \in B$. Therefore, we get (6). Similarly, the fact that Δ_t^1 is an algebra morphism (resp., Δ_t^1 is a coassociative comultiplication) gives (7) (resp., (8)), i.e., (f, g) can be viewed as an element in $\hat{h}_b^2(B)_{(-1)}$.

Suppose that $(B[t]/(t^2), m_t^1, \Delta_t^1)$ and $(B[t]/(t^2), {m'}_t^1, {\Delta'}_t^1)$ are two isomorphic deformations, with (f, g) and (f', g') defined as above. Let ϕ be the isomorphism. We may write

$$\phi(a) = a + \theta(a)t, \quad a \in B$$

for some homogeneous map $\theta : B \to B$ of degree -1 (note that the map θ may be viewed as a map from m to m). Now it is direct to check that θ realizes an equivalence of (f, g) and (f', g') in $\hat{h}_b^2(B)_{(-1)}$. Hence, we have obtained a map from $\mathcal{E}^1(B)$ to $\hat{h}_b^2(B)_{(-1)}$ sending $(B[t]/(t^2), m_t^1, \Delta_t^1)$ to (f, g). One can easily see that the correspondence is bijective, as required.

(ii) To prove that B is graded-rigid, we just need to show that $\operatorname{iso}\mathcal{E}(B)$ is a single-element set.

Let $(B[t], m_t, \Delta_t)$ be an element in $\mathcal{E}(B)$. As before, write

$$m_t(a \otimes b) = \sum_{s=0}^{\infty} m_s(a \otimes b)t^s \quad \text{and} \quad \Delta_t(c) = \sum_{s=0}^{\infty} \Delta_s(c)t^s.$$

Note that $m_0 = m$, $\Delta_0 = \Delta$, and m_s, Δ_s are homogeneous maps of degree $-s$. By a similar argument as (i), we may view $m_s \in D^{1,2}_{(-s)}$ and $\Delta_s \in D^{2,1}_{(-s)}$. Moreover, from (i), we see that (m_1, Δ_1) can be viewed as an element in $\hat{h}_b^2(B)_{(-1)}$. Now by assumption, there exists a homogeneous map $\theta_1 : \mathsf{m} \to \mathsf{m}$ of degree -1 such that (see (9) and (10)):

$$\begin{aligned} m_1(a \otimes b) &= a\theta_1(b) - \theta_1(ab) + \theta_1(a)b, \\ \Delta_1(c) &= \Delta(\theta_1(c)) - c_{(1)} \otimes \theta_1(c_{(2)}) - \theta_1(c_{(1)}) \otimes c_{(2)}. \end{aligned}$$

Take $\phi_1 : B[t] \to B[t]$ to be a $\mathbb{K}[t]$-linear map such that

$$\phi_1(a) = a + \theta_1(a)t, \quad a \in B.$$

Note that ϕ_1 is a bijective map preserving the identity 1_B and counit ε_t. Consider the deformation

$$(B[t],\ {m_t}' = \phi_1 \circ m_t \circ (\phi_1^{-1} \otimes \phi_1^{-1}),\ {\Delta_t}' = (\phi_1 \otimes \phi_1) \circ \Delta_t \circ \phi_1^{-1}).$$

We have

$$\begin{aligned} {m_t}'(a\otimes b) &= ab + {m_2}'(a\otimes b)t^2 + {m_3}'(a\otimes b)t^3 + \cdots, \\ {\Delta_t}'(c) &= \Delta(c) + {\Delta_2}'(c)t^2 + {\Delta_2}'(c)t^3 + \cdots, \end{aligned}$$

where ${m_s}'$ and ${\Delta_s}'$ are homogeneous maps of degree $-s$, $s \geq 2$. Now by comparing (3)–(5) with (6)–(8), we see that $({m_2}', {\Delta_2}')$ can be viewed as an element in $\hat{h}^2_b(B)_{(-2)}$. Hence, there exists a homogeneous map $\theta_2 : \mathsf{m} \to \mathsf{m}$ of degree -2 such that (again see (9) and (10))

$$\begin{aligned} {m_2}'(a\otimes b) &= a\theta_2(b) - \theta_2(ab) + \theta_2(a)b, \\ {\Delta_2}'(c) &= \Delta(\theta_2(c)) - c_{(1)} \otimes \theta_2(c_{(2)}) - \theta_2(c_{(1)}) \otimes c_{(2)}. \end{aligned}$$

Take $\phi_2 : B[t] \to B[t]$ to be a $\mathbb{K}[t]$-linear map such that

$$\phi_2(a) = a + \theta_2(a)t^2, \quad a \in B.$$

Now consider the following deformation

$$(B[t],\ {m_t}'' = \phi_2 \circ {m_t}' \circ (\phi_2^{-1} \otimes \phi_2^{-1}),\ {\Delta_t}'' = (\phi_2 \otimes \phi_2) \circ {\Delta_t}' \circ \phi_2^{-1}),$$

whose coefficients of t and t^2 vanish. In other words,

$$\begin{aligned} {m_t}''(a\otimes b) &= ab + {m_3}''(a\otimes b)t^3 + {m_3}''(a\otimes b)t^4 + \cdots, \\ {\Delta_t}''(c) &= \Delta(c) + {\Delta_3}''(c)t^3 + {\Delta_2}''(c)t^4 + \cdots. \end{aligned}$$

Similarly, we may view $({m_3}'', {\Delta_3}'')$ as in $\hat{h}^2_b(B)_{(-3)}$. By assumption and comparing (6)–(8), we have a homogeneous map $\theta_3 : \mathsf{m} \to \mathsf{m}$ such that

$$\begin{aligned} {m_3}''(a\otimes b) &= a\theta_3(b) - \theta_3(ab) + \theta_3(a)b, \\ {\Delta_3}''(c) &= \Delta(\theta_3(c)) - c_{(1)} \otimes \theta_3(c_{(2)}) - \theta_3(c_{(1)}) \otimes c_{(2)}. \end{aligned}$$

Now define $\phi_3 : B[t] \to B[t]$ to be a $\mathbb{K}[t]$-linear map such that

$$\phi_3(a) = a + \theta_3(a)t^3, \quad a \in B.$$

Thus, we get the following deformation

$$(B[t],\ {m_t}''' = \phi_3 \circ {m_t}'' \circ (\phi_3^{-1} \otimes \phi_3^{-1}),\ {\Delta_t}''' = (\phi_3 \otimes \phi_3) \circ {\Delta_t}'' \circ \phi_3^{-1}),$$

whose coefficients of t, t^2 and t^3 vanish. Now one can define θ_4 and ϕ_4, and so on.

Finally, define the infinite composition $\cdots \circ \phi_3 \circ \phi_2 \circ \phi_1$ as ϕ. Note that the $\mathbb{K}[t]$-linear isomorphism $\phi : B[t] \to B[t]$ is well defined for every $a \in B$, which preserves the identity 1_B and counit ε_t. In fact, $\phi_s(a) = a + \theta_s(a)t^s$, where $\theta_s : \mathsf{m} \to \mathsf{m}$ is homogeneous of degree $-s$, hence for each fixed $a \in B_{(i)}$, $\phi_s(a) = a$ for $s \geq i$. Consequently, $\phi(a)$ has nonzero coefficients of t^s only for $0 \leq s \leq i$. By the construction of each map ϕ_s, we obtain that the deformation

$$(B[t],\ \phi \circ m_t \circ (\phi^{-1} \otimes \phi^{-1}),\ (\phi \otimes \phi) \circ \Delta_t \circ \phi^{-1})$$

is trivial, which is also equivalent to the given deformation. Thus, (ii) is proved.

(iii) Let $(B[t]/(t^{l+1}), m_t^l, \Delta_t^l)$ be an element in $\mathcal{E}^l(B)$. Write

$$m_t^l(a\otimes b) = \sum_{0\le s\le l} m_s(a\otimes b)t^s \quad \text{and} \quad \Delta_t^l(c) = \sum_{0\le s\le l} \Delta_s(c)t^s,$$

where m_s and Δ_s are homogeneous maps of degree $-s$. By the same argument as above, one can show that m_s (resp., Δ_s) can be viewed as a map from $\mathsf{m}\otimes\mathsf{m}$ to m (resp., from m to $\mathsf{m}\otimes\mathsf{m}$).

To extend $(B[t]/(t^{l+1}), m_t^l, \Delta_t^l)$ to some element in $\mathcal{E}^{l+1}(B)$, we just need to find some homogeneous maps $f:\mathsf{m}\otimes\mathsf{m}\to\mathsf{m}$ and $g:\mathsf{m}\to\mathsf{m}\otimes\mathsf{m}$ of degree $-(l+1)$ such that $(B[t]/(t^{l+2}), m_t^l + t^{l+1}f, \Delta_t^l + t^{l+1}g)$ is a bialgebra over $\mathbb{K}[t]/(t^{l+2})$.

The associativity of $m_t^l + t^{l+1}f$ is equivalent to

$$\begin{aligned} &(m_t^l + t^{l+1}f)\big(((m_t^l + t^{l+1}f)(a\otimes b))\otimes c\big)\\ =\ &(m_t^l + t^{l+1}f)\big(a\otimes((m_t^l + t^{l+1}f)(b\otimes c))\big)\end{aligned}$$

for all $a,b,c\in B$. Since m_t^l is associative, the above identity holds if and only if the two sides have the same coefficient of the term t^{l+1}. Thus, by direct computation, we get

$$\begin{aligned} F(a\otimes b\otimes c) :&= \sum_{s=1}^{l} m_s(m_{l+1-s}(a\otimes b)\otimes c) - m_s(a\otimes m_{l+1-s}(b\otimes c))\\ &= af(b\otimes c) - f(ab\otimes c) + f(a\otimes bc) - f(a\otimes b)c\\ &= \delta_h^{1,2}(f)(a\otimes b\otimes c).\end{aligned}$$

Similarly, the compatibility of the multiplication $m_t^l + t^{l+1}f$ and comultiplication $\Delta_t^l + t^{l+1}g$, and the coassociativity of $\Delta_t^l + t^{l+1}g$ are respectively equivalent to the following two identities:

$$\begin{aligned} H(a\otimes b) :&= \sum_{s=1}^{l} \Delta_s(m_{l+1-s}(a\otimes b)) - \sum_{s+r+s'+r'=l+1} (m_{s'}\otimes m_{r'})\circ\tau_{23}(\Delta_s(a)\otimes\Delta_r(b))\\ &= (\delta_c^{1,2}(f) + \delta_h^{2,1}(g))(a\otimes b),\end{aligned}$$

$$\begin{aligned} G(c) :&= \sum_{s=1}^{l} (\Delta_s\otimes \mathrm{Id})\circ\Delta_{l+1-s}(c) - (\mathrm{Id}\otimes\Delta_s)\circ\Delta_{l+1-s}(c)\\ &= c_{(1)}\otimes g(c_{(2)}) - (\Delta\otimes\mathrm{Id})(g(c)) + (\mathrm{Id}\otimes\Delta)(g(c)) - g(c_{(1)})\otimes c_{(2)}\\ &= \delta_c^{2,1}(g)(c),\end{aligned}$$

where $a,b,c\in\mathsf{m}$, and τ_{23} is the flip map with respect to the second and third positions.

It is direct to check that the element $(F,H,G)\in\hat{D}_{(-l-1)}$ is a cocycle (exactly as [4] in the case of algebras and [6] in the case of non-graded bialgebras), i.e., it lies in the kernel of the differential $\partial^3_{(-l-1)}$ from (3)–(5), therefore, it can be viewed as an element in the cohomology group $\hat{h}_b^3(B)_{(-l-1)}$. By comparing the above three identities with (11), we obtain that if $\hat{h}_b^3(B)_{(-l-1)} = 0$, then such maps f and g always exist, i.e., we can extend $(B[t]/(t^{l+1}), m_t^l, \Delta_t^l)$ to

$$(B[t]/(t^{l+2}),\, m_t^l + t^{l+1}f,\, \Delta_t^l + t^{l+1}g),$$

which lies in $\mathcal{E}^{l+1}(B)$ by the above three equivalences. This completes the proof. □

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